Senior INTERMEDIATE

Maths-IIB

Calculus

★ Area bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}$ ab. S.U

★ Area bounded by $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}$ a^2 S.U

★ Area bounded by $y^2 = 24x$ and $x^2 = 12y$ is $\frac{16}{3}(6)(3)$ S.U

★ Area bounded by $y^2 = 12x$ and $x^2 = 8y$ is $\frac{16}{3}(3)(2)$ S.U

★ Area bounded by $y^2 = 8x$ and $x^2 = 4y$ is $\frac{16}{3}(2)(1)$ S.U

★ Area bounded by $y^2 = 4x$ and $x^2 = 4y$ is $\frac{16}{3}(1)(1)$ S.U

★ $I = \int \frac{dx}{1 + \sin x + \cos x}$

$= \int \frac{dx}{\frac{1 + 2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}}$

$= \int \frac{1 + \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} .dx$

$= \int \frac{\sec^2\frac{x}{2} . dx}{2\left(1 + \tan\frac{x}{2}\right)}$

$= \int \frac{\frac{1}{2} \cdot \sec^2\frac{x}{2} . dx}{1 + \tan\frac{x}{2}}$

$= \log\left|1 + \tan\frac{x}{2}\right| + c$

$\because \frac{d}{dx}\left(1 + \tan\frac{x}{2}\right) = \frac{1}{2}\sec^2\frac{x}{2}$

★ $\int \frac{\sec x . dx}{\sec x + \cos x}$

$= \int \frac{dx}{1 + \cos^2 x}$

$= \int \frac{dx}{1 + \frac{1 + \cos 2x}{2}}$

$= \int \frac{2}{3 + \cos 2x} dx$

$= \int \frac{2}{3 + \frac{1 - \tan^2 x}{1 + \tan^2 x}} dx$

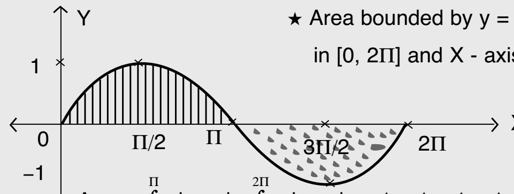
$= \int \frac{2\sec^2 x .dx}{4 + 2\tan^2 x}$

$= \int \frac{\sec^2 x .dx}{2 + \tan^2 x}$

$= \int \frac{\sec^2 x .dx}{(\sqrt{2})^2 + (\tan x)^2}$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + c$

$\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$



★ Area bounded by y = sin x in [0, 2Π] and X - axis is 4 S.U.

Area $= \int_0^{\Pi} \sin x\, dx - \int_0^{2\Pi} \sin x\, dx = 1 + 1 + 1 + 1 = 4$ S.U.

Fixed Quantity through Integration...

★ $\int_0^{\pi/4} \frac{\sec x . dx}{\sec x + \tan x}$

$= \int_0^{\pi/4} \frac{dx}{1 + \sin x}$

$= \int_0^{\pi/4} \left(\frac{1 - \sin x}{\cos^2 x}\right) dx$

$= \int_0^{\pi/4} \sec^2 x\, dx - \int_0^{\pi/4} \sec x \tan x\, dx$

$= \left(\tan x - \sec x\right)_0^{\pi/4}$

$= \left(1 - \sqrt{2}\right) - \left(0 - 1\right) = 2 - \sqrt{2}$

★ $I_{m, n} = \int \sin^m x . \cos^n x\, dx$

$= \int \sin^{m-1} x . \sin x . \cos^n x\, dx$

$= \left(\sin^{m-1} x . \int \sin x . \cos^n x\, dx\right)$

$- \int \left[\frac{d}{dx}\left(\sin^{m-1} x\right) . \int \sin x . \cos^n x\, dx\right] dx$

$= \sin^{m-1} x \left(\frac{-\cos^{n+1} x}{n+1}\right)$

$- \int (m-1) \sin^{m-2} x . \cos x \left(\frac{-\cos^{n+1} x}{n+1}\right) dx$

$= \frac{-\sin^{m-1} x . \cos^{n+1} x}{n+1}$

$+ \frac{m-1}{n+1} . \int \sin^{m-2} x . \cos^{n+2} x . dx$

$= \frac{-\sin^{m-1} x . \cos^{n+1} x}{n+1}$

$+ \frac{m-1}{n+1} \int \sin^{m-2} x \left(1 - \sin^2 x\right) . \cos^n x\, dx$

$= \frac{-\sin^{m-1} x \cdot \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x$

$. \cos^n x\, dx - \frac{m-1}{n+1} \int \sin^m x . \cos^n x\, dx$

$I_{m, n} = \frac{-\sin^{m-1} x . \cos^{n+1} x}{n+1}$

$+ \frac{m-1}{n+1} . I_{m-2, n} - \frac{m-1}{n+1} . I_{m,n}$

$\left(1 + \frac{m-1}{n+1}\right) I_{m,n}$

$= \frac{-\sin^{m-1} x . \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} . I_{m-2, n}$

$\therefore I_{m, n}$

$= \frac{-\sin^{m-1} x . \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} . I_{m-2, n}$

★ $\int_0^{\pi/2} \sin^m x . \cos^n x . dx$

$= \left(\frac{-\sin^{m-1} x . \cos^{n+1} x}{m+n}\right)_0^{\pi/2} + \frac{m-1}{m+n} . I_{m-2, n}$

$\therefore I_{m, n} = \frac{m-1}{m+n} \cdot I_{m-2, n}$

Examples

1) $I_{5, 3} = \frac{4}{8} \cdot \frac{2}{6} \cdot I_{1, 3}$

$= \frac{4}{8} \cdot \frac{2}{6} . \int_0^{\pi/2} \sin x . \cos^3 x . dx$

$= \frac{4}{8} \cdot \frac{2}{6} \cdot \left(-\frac{\cos^4 x}{4}\right)_0^{\pi/2}$

$= \frac{4}{8} \cdot \frac{2}{6} \cdot \frac{1}{4}$

2) $I_{5, 4} = \frac{4}{9} \cdot \frac{2}{7} \cdot I_{1, 4}$

$= \frac{4}{9} \cdot \frac{2}{7} . \int_0^{\pi/2} \sin x . \cos^4 x . dx$

$= \frac{4}{9} \cdot \frac{2}{7} \cdot \left(-\frac{\cos^5 x}{5}\right)_0^{\pi/2}$

$= \frac{4}{9} \cdot \frac{2}{7} \cdot \frac{1}{5}$

3) $I_{6, 5} = \frac{5}{11} \cdot \frac{3}{9} \cdot \frac{1}{7} \cdot I_{0, 5}$

$= \frac{5}{11} \cdot \frac{3}{9} \cdot \frac{1}{7} \cdot \int_0^{\pi/2} \cos^5 x . dx$

$= \frac{5}{11} \cdot \frac{3}{9} \cdot \frac{1}{7} \cdot \left(\frac{4 . 2}{5 . 3} \times 1\right)$

$= \frac{5 . 3 . 1 . 4 . 2 . 1}{11 . 9 . 7 . 5 . 3}$

4) $I_{6, 6} = \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} \cdot I_{0, 6}$

$= \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} \cdot \int_0^{\pi/2} \cos^6 x . dx$

$= \frac{5}{12} \cdot \frac{3}{10} \cdot \frac{1}{8} \cdot \left(\frac{5 . 3 . 1}{6 . 4 . 2} \times \frac{\pi}{2}\right)$

$= \frac{5 . 3 . 1 . 5 . 3 . 1}{12 . 10 . 8 . 6 . 4 . 2} \times \frac{\pi}{2}$

★ $\int_0^4 |x - 2|\, dx$

$= \int_0^2 |x - 2|\, dx + \int_2^4 |x - 2|\, dx$

$= -\int_0^2 (x - 2)\, dx + \int_2^4 (x - 2) dx$

$= \left(2x - \frac{x^2}{2}\right)_0^2 + \left(\frac{x^2}{2} - 2x\right)_2^4$

$= 2 + 2 = 4$

Q: Solve: $(x + 2y - 1)\, dy = (2x + y - 1)\, dx$

A: $\frac{dy}{dx} = \frac{2x + y - 1}{x + 2y - 1}$

For solving this, put x = x + h; y = y + k

$\therefore$ We have

$\frac{dy}{dx} = \frac{(2x + y) + (2h + k - 1)}{(x + 2y) + (h + 2k - 1)}$

For converting this into homogeneous equation put 2h + k – 1 = 0 and h + 2k – 1 = 0

$\therefore$ we have $h = \frac{1}{3}$; $k = \frac{1}{3}$

$\therefore \frac{dy}{dx} = \frac{2x + y}{x + 2y}$

$\because$ Homogeneous equation in x and y,

take y = vx (or) $\frac{y}{x} = v$

then $\frac{dy}{dx} = v + x . \frac{dv}{dx}$

now, $v + x . \frac{dv}{dx} = \frac{2x + vx}{x + 2vx} = \frac{2 + v}{1 + 2v}$

$x . \frac{dv}{dx} = \frac{2 + v}{1 + 2v} - \frac{v}{1}$

$= \frac{2 + v - v - 2v^2}{1 + 2v}$

$= \frac{2(1 - v^2)}{1 + 2v}$

$\therefore \left(\frac{1 + 2v}{1 - v^2}\right) dv = 2 . \frac{dy}{x}$

Integrating both sides,

$\int \frac{1}{1 - v^2} dv + \int \frac{2v}{1 - v^2} dv = 2\int \frac{1}{x} dx$

$\Rightarrow \frac{1}{2(1)} \log\left|\frac{1 + v}{1 - v}\right| - \log\left|1 - v^2\right|$

$= 2\log|x| + \log|c|$

$\Rightarrow \frac{1}{2} . \log\left|\frac{1 + v}{1 - v}\right| = \log\left|x^2 . c . (1 - v^2)\right|$

$\Rightarrow \log\left|\frac{1 + v}{1 - v}\right| = \log\left|x^2 . c . (1 - v^2)\right|^2$

$\Rightarrow \frac{1 + v}{1 - v} = \left[x^2 . c . (1 - v^2)\right]^2$

$\Rightarrow \frac{x + y}{x - y} = x^4 . c^2 . \left(\frac{x^2 + y^2}{x^2}\right)^2$

$\because v = \frac{y}{x}$

$\Rightarrow \frac{x + y}{x - y} = c^2 . \left(x^2 - y^2\right)^2$

$\Rightarrow \frac{1}{c^2} = (x + y)(x - y)^3$

$\Rightarrow k^2 = \left(x + y - \frac{2}{3}\right)\left(x - y\right)^3$

$\Rightarrow 3k^2 = (3x + 3y - 2)(x - y)^3$

$\therefore$ Solution:

$(3x + 3y - 2)(x - y)^3 = k$, a constant.

★ $I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} . I_{n-2}$

Ex: $I_6 = \int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

★ $I_n = \int_0^{\pi/2k} \sin^n kx\, dx = \frac{1}{k} \cdot \frac{n-1}{n} \cdot I_{n-2}$

Ex: $I_6 = \int_0^{\pi/8} \sin^6 4x .dx = \frac{1}{4}\left(\frac{5 . 3 . 1}{6 . 4 . 2}\right) \cdot \frac{\pi}{2}$